Hypercontractivity and Its Applications for Functional SDEs of Neutral Type *

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Abstract

In this paper, we discuss hypercontractivity for the Markov semigroup P_t which is generated by segment processes associated with a range of functional SDEs of neutral type. As applications, we also reveal that the semigroup P_t converges exponentially to its unique invariant probability measure μ in entropy, $L^2(\mu)$ and $\|\cdot\|_{\rm var}$, respectively.

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1 Introduction

According to [9] by Gross, the Markov semigroup P_t is called hypercontractivity with respect to the invariant probability measure μ if $\|P_t\|_{2\to4} \le 1$ for large $t > 0$, where $\|\cdot\|_{2\to4}$ stands for the operator norm from $L^2(\mu)$ to $L^4(\mu)$. The hypercontractivity of Markov semigroups has been extensively studied for various models (see, e.g., [1, 2, 7, 9, 16, 18, 19, 20] and references therein). In the light of [9], the log-Sobolev inequality implies the hypercontractivity. However, the approach adopted in [9] no longer works for functional SDEs since the log-Sobolev inequality for the associated Dirichlet form is invalid. In [2], utilizing the Harnack inequality with power initiated in [15], the authors investigated the hypercontractivity and its applications for a range of non-degenerate functional SDEs. Wang [20] developed a general framework on how to establish the hypercontractivity for Markov semigroups (see [20, Theorem 2.1]). Meanwhile, he applied successfully the theory to finite/infinite dimensional stochastic Hamiltonian systems.

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In this paper, as a continuation of our work [2], we are still interested in the hypercontractivity and its applications, however, for a kind of functional SDEs of *neutral type*. For a functional SDE of neutral type, we mean an SDE which not only depends on the past and the present values but also involves derivatives with delays (see, e.g., [11, Chapter 6]). Such equation has also been utilized to model some evolution phenomena arising in, e.g., physics, biology and engineering. See, e.g., Kolmanovskii-Nosov [10] concerning the theory in aeroelasticity, Mao [11] with regard to the collision problem in electrodynamics, and Slemrod [13] for the oscillatory systems, to name a few. For large deviation of functional SDEs of neutral type, we refer to Bao-Yuan [5].

We introduce some notation. Let $(\mathbb{R}^n, \langle\cdot,\cdot\rangle, |\cdot|)$ be an n-dimensional Euclidean space, and $\{W(t)\}_{t\geq 0}$ an n-dimensional Brownian motion defined on a complete filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$. For a closed interval $I \subset \mathbb{R}$, $C(I;\mathbb{R}^n)$ denotes the collection of all continuous functions $f : I \mapsto \mathbb{R}^n$. For a fixed constant $r_0 > 0$, let $\mathscr{C} := C([-r_0, 0];\mathbb{R}^n)$ endowed with the uniform norm $\|f\|_\infty := \sup_{-r_0\leq\theta\leq 0} |f(\theta)|$ for $f \in \mathscr{C}$. For $X(\cdot) \in C([-r_0,\infty);\mathbb{R}^n)$ and $t \geq 0$, define the segment process $X_t \in \mathscr{C}$ by $X_t(\theta) := X(t+\theta)$, $\theta \in [-r_0, 0]$. $\mathbb{R}^n \otimes \mathbb{R}^n$ means the family of all $n \times n$ matrices. $\mathscr{P}(\mathscr{C})$ stands for the set of all probability measures on $\mathscr{C}$, and $\|\cdot\|_{\mathrm{var}}$ denotes the total variation norm.

In the paper, we focus on a functional SDE of neutral type in the framework

$$\mathrm{d}\{X(t) + LX_t\} = \{Z(X(t)) + b(X_t)\}\mathrm{d}t + \sigma\mathrm{d}W(t), \quad t > 0, \quad X_0 = \xi \in \mathscr{C}, \tag{1.1}$$

where, for any $\phi \in \mathscr{C}$,

$$L\phi := \kappa \int_{-r_0}^{0} \phi(\theta)\mathrm{d}\theta, \text{ with } \kappa \in (0,1), \tag{1.2}$$

$Z : \mathbb{R}^n \mapsto \mathbb{R}^n$ and $b : \mathscr{C} \mapsto \mathbb{R}^n$ are progressively measurable, and $\sigma \in \mathbb{R}^n \otimes \mathbb{R}^n$ is an invertible matrix.

Throughout the paper, for any $x, y \in \mathbb{R}^n$ and $\xi, \eta \in \mathscr{C}$, we assume that

(H1) Z and b are Lipschitzian with Lipschitz constants $L_1 > 0$ and $L_2 > 0$, respectively, i.e., $|Z(x) - Z(y)| \leq L_1|x - y|$ and $|b(\xi) - b(\eta)| \leq L_2\|\xi - \eta\|_\infty$;

(H2) There exist constants $\lambda_1 > \lambda_2 > 0$ such that

$$2\langle Z(\xi(0))-Z(\eta(0))+b(\xi)-b(\eta), \xi(0)-\eta(0)+L(\xi-\eta)\rangle \leq \lambda_2\|\xi-\eta\|_\infty^2-\lambda_1|\xi(0)-\eta(0)|^2.$$

On the basis of (H1) and $\kappa \in (0,1)$, (1.1) admits a unique strong solution $\{X(t)\}_{t\geq -r_0}$ (see, e.g., [11, Theorem 2.2, p204]). The dissipative-type condition (H2) is imposed to reveal the long-time behavior of segment process $\{X_t\}_{t\geq 0}$. For more details, please refer to Lemmas 2.2-2.4 below. Moreover, for illustrative examples such that (H2) holds, we would like to refer to, e.g., [12, Example 5.2]. On occasions, to emphasize the initial datum $X_0 = \xi \in \mathscr{C}$, we write $\{X(t;\xi)\}_{t\geq -r_0}$ instead of $\{X(t)\}_{t\geq -r_0}$, and $\{X_t(\xi)\}_{t\geq 0}$ in lieu of $\{X_t\}_{t\geq 0}$, respectively.

As we present in the beginning of the second paragraph, in this paper, we intend to investigate the hypercontractivity and its applications for the Markov semigroup

$$P_tf(\xi) := \mathbb{E}f(X_t(\xi)),\ f \in \mathscr{B}_b(\mathscr{C}),\ \xi \in \mathscr{C}. \tag{1.3}$$

Our main result in this paper is stated as below.

Theorem 1.1. Let (H1)-(H2) hold and assume $\kappa \in (0,1)$, and suppose further that

$$1-\kappa r_0^2 \mathrm{e}^{\rho r_0} > 0 \quad \text{and} \quad \lambda := \rho - \frac{(\kappa r_0^2\lambda_1+\lambda_2)\mathrm{e}^{\rho r_0}}{(1-\kappa)(1-\kappa r_0^2\mathrm{e}^{\rho r_0})} > 0. \tag{1.4}$$

with $\rho = \lambda_1/(1+\kappa)$. Then, the following assertions hold.

(1) P_t has a unique invariant probability measure μ.

(2) P_t is hypercontractive.

(3) P_t is compact on $L^2(\mu)$ for large enough $t > 0$, and there exist $c, \alpha > 0$ such that

$$\mu((P_tf)\log P_tf) \le c\mathrm{e}^{-\alpha t}\mu(f\log f)$$

(4) There exists a constant $C > 0$ such that

$$\|P_t-\mu\|_2^2 := \sup_{\mu(f^2)\le 1} \mu\big((P_tf-\mu(f))^2\big) \le C\mathrm{e}^{-\lambda t}, \quad t \ge 0.$$

(5) There exist two constants $t_0, C > 0$ such that

$$\|P_t^\xi - P_t^\eta\|_{\mathrm{var}}^2 \le C\|\xi-\eta\|_\infty^2 \mathrm{e}^{-\lambda t}, \quad t \ge t_0,$$

where P_t^ξ stands for the law of X_t^ξ for $(t,\xi) \in [0,\infty)\times\mathscr{C}$.

Compared with [2, 14, 20], this paper contains the following new points: (1) Theorem 1.1 works for functional SDEs of *neutral type* and covers [2, Theorem 1.1] whenever $\kappa = 0$; (2) The argument of Lemma 2.5 gives some new ideas on how to establish the dimensional-free Harnack inequality for functional SDEs of neutral type. The remainder of this paper is organized as follows. In Section 2, we investigate Gauss-type concentration property (see Lemma 2.2), existence and uniqueness of invariant measure (see Lemma 2.4), and the Harnack inequality (see Lemma 2.5), and devote to completing the proof of Theorem 1.1.

2 Proof of Theorem 1.1

[20, Theorem 2.1], which is stated as Lemma 2.1 below for presentation convenience, establishes a general result on the hypercontractivity of Markov semigroups. For the present situation, the key point in the proof of Theorem 1.1 is to realize (i)-(iii) in [20, Theorem 2.1], one by one.

Let $(E, \mathscr{B}, \mu)$ be a probability space, and P_t a Markov semigroup on $\mathscr{B}_b(E)$ such that μ is P_t-invariant. Recall that a process (X_t, Y_t) on $E \times E$ is called a coupling associated with the semigroup P_t if

$$P_t f(\xi) = \mathbb{E}(f(X_t)|\xi), \quad P_t f(\eta) = \mathbb{E}(f(Y_t)|\eta), \quad f \in \mathscr{B}_b(E), \ t \geq 0.$$

Lemma 2.1. Assume that the following three conditions hold for some measurable functions $\rho : E \times E \mapsto (0, \infty)$ and $\phi : [0, \infty) \mapsto (0, \infty)$ with $\lim_{t\to\infty} \phi(t) = 0$:

(i) There exist constants $t_0, c_0 > 0$ such that

$$(P_{t_0} f(\xi))^2 \leq (P_{t_0} f^2(\eta))\mathrm{e}^{c_0\rho(\xi,\eta)^2}, \quad f \in \mathscr{B}_b(E), \ \xi, \eta \in E;$$

(ii) For any $\xi, \eta \in E \times E$, there exists a coupling X_t, Y_t associated with P_t such that

$$\rho(X_t, Y_t) \leq \phi(t)\rho(\xi, \eta), \quad t \geq 0;$$

(iii) There exists $\varepsilon > 0$ such that $(\mu \times \mu)(\mathrm{e}^{\varepsilon\rho^2}) < \infty$.

Then μ is the unique invariant probability measure of P_t, P_t is hypercontractive and compact in $L^2(\mu)$ for large $t > 0$.

Hereinafter, we first investigate the following exponential-type estimate, which plays a crucial role in discussing the hypercontractivity.

Lemma 2.2. Under the assumptions of Theorem 1.1, there exist $\varepsilon, c > 0$ such that

$$\mathbb{E}\,\mathrm{e}^{\varepsilon\|X_t^\xi\|_\infty^2} \leq \mathrm{e}^{c(1+\|\xi\|_\infty^2)}, \quad t \geq 0, \ \xi \in \mathscr{C}. \tag{2.1}$$

Proof. For $\eta(\theta) \equiv 0, \theta \in [-r_0, 0]$, by (H2) and (1.4), we deduce that

$$\begin{aligned} &2\langle \xi(0) + L\xi, Z(\xi(0)) + b(\xi)\rangle \\ &= 2\langle \xi(0) - \eta(0) + L(\xi - \eta), Z(\xi(0)) - Z(\eta(0)) + b(\xi) - b(\eta) + Z(\eta(0)) + b(\eta)\rangle \\ &\leq c_0 - \lambda_1'|\xi(0)|^2 + \lambda_2'\|\xi\|_\infty^2 \end{aligned} \tag{2.2}$$

for some constants $c_0 > 0$ and $\lambda_1', \lambda_2' > 0$ such that

$$1 - \kappa r_0^2 \mathrm{e}^{\rho' r_0} > 0, \qquad \lambda' := \rho' - \frac{(\kappa r_0^2 \lambda_1' + \lambda_2')\mathrm{e}^{\rho' r_0}}{(1-\kappa)(1 - \kappa r_0^2 \mathrm{e}^{\rho' r_0})} > 0, \tag{2.3}$$

where $\rho' = \lambda_1'/(1+\kappa)$. For simplicity, let $\Gamma(t) := X(t) + LX_t$. By Itô's formula, it follows from (2.2) that

$$\mathrm{d}\{\mathrm{e}^{\rho' t}|\Gamma(t)|^2\} \le \mathrm{e}^{\rho' t}\{\rho'|\Gamma(t)|^2 + c_0 - \lambda_1'|X(t)|^2 + \lambda_2'\|X_t\|_\infty^2\}\mathrm{d}t + \mathrm{d}M(t), \tag{2.4}$$

where $\mathrm{d}M(t) := 2\mathrm{e}^{\rho' t}\langle\Gamma(t), \sigma\mathrm{d}W(t)\rangle$. Recall the following elementary inequality:

$$(a+b)^2 \le (1+\delta)a^2 + (1+\delta^{-1})b^2,\ \ a, b \in \mathbb{R},\ \ \delta > 0. \tag{2.5}$$

So one has

$$|\Gamma(t)|^2 \le (1+\kappa)|X(t)|^2 + (1+\kappa^{-1})|LX_t|^2,$$

which, together with $|L\phi| \le \kappa\|\phi\|_\infty$ for $\phi \in \mathscr{C}$ and $\kappa \in (0,1)$, leads to

$$-|X(t)|^2 \le -\frac{1}{1+\kappa}|\Gamma(t)|^2 + \kappa r_0^2\|X_t\|_\infty^2.$$

Substituting this into (2.4) gives that

$$\mathrm{e}^{\rho' t}|\Gamma(t)|^2 \le |\Gamma(0)|^2 + \frac{c_0}{\rho'}\mathrm{e}^{\rho' t} + N(t) + (\kappa r_0^2\lambda_1' + \lambda_2')\int_0^t \mathrm{e}^{\rho' s}\|X_s\|_\infty^2\mathrm{d}s, \tag{2.6}$$

where $N(t) := \sup_{0\le s\le t} M(s)$. Choosing $\delta = \frac{\kappa}{1-\kappa}$ in (2.5) and noting $|L\phi| \le \kappa r_0\|\phi\|_\infty$ for $\phi \in \mathscr{C}$ and $\kappa \in (0,1)$ into consideration, we derive that

$$\begin{aligned}\mathrm{e}^{\rho' s}|X(s)|^2 &\le \frac{1}{1-\kappa}\mathrm{e}^{\rho' s}|\Gamma(s)|^2 + \kappa r_0^2\mathrm{e}^{\rho' r_0}\sup_{-r_0\le\theta\le 0}(\mathrm{e}^{\rho'(s+\theta)}|X(s+\theta)|^2)\\ &\le \frac{1}{1-\kappa}\mathrm{e}^{\rho' s}|\Gamma(s)|^2 + \kappa r_0^2\mathrm{e}^{\rho' r_0}\sup_{s-r_0\le u\le s}(\mathrm{e}^{\rho' u}|X(u)|^2).\end{aligned}$$

We then have

$$\sup_{0\le s\le t}(\mathrm{e}^{\rho' s}|X(s)|^2) \le \kappa r_0^2\mathrm{e}^{\rho' r_0}\|\xi\|_\infty^2 + \frac{1}{1-\kappa}\sup_{0\le s\le t}(\mathrm{e}^{\rho' s}|\Gamma(s)|^2) + \kappa r_0^2\mathrm{e}^{\rho' r_0}\sup_{0\le s\le t}(\mathrm{e}^{\rho' s}|X(s)|^2)$$

so that, due to (2.3),

$$\sup_{0\le s\le t}(\mathrm{e}^{\rho' s}|X(s)|^2) \le \frac{\kappa r_0^2\mathrm{e}^{\rho' r_0}}{1-\kappa r_0^2\mathrm{e}^{\rho' r_0}}\|\xi\|_\infty^2 + \lambda''\sup_{0\le s\le t}(\mathrm{e}^{\rho' s}|\Gamma(s)|^2), \tag{2.7}$$

in which $\lambda'' := (1-\kappa)^{-1}(1-\kappa r_0^2\mathrm{e}^{\rho' r_0})^{-1}$. Moreover, it is easy to see that

$$\mathrm{e}^{\rho' t}\|X_t\|_\infty^2 \le \mathrm{e}^{\rho' r_0}\sup_{t-r_0\le s\le t}(\mathrm{e}^{\rho' s}|X(s)|^2). \tag{2.8}$$

Thus, from (2.7) and (2.8), we infer that

$$\mathrm{e}^{\rho' t}\|X_t\|_\infty^2 \le c_1\|\xi\|_\infty^2 + \lambda''\mathrm{e}^{\rho' r_0}\sup_{0\le s\le t}(\mathrm{e}^{\rho' s}|\Gamma(s)|^2) \tag{2.9}$$

for some $c_1 > 0$. Taking (2.6) and (2.9) into account yields that

$$\mathrm{e}^{\rho' t}\|X_t\|_\infty^2 \le c_1\|\xi\|_\infty^2 + c_2(N(t)+\mathrm{e}^{\rho' t}) + \gamma' \int_0^t \mathrm{e}^{\rho' s}\|X_s\|_\infty^2 \mathrm{d}s \tag{2.10}$$

for some $c_2 > 0$, where $\gamma' := \lambda'' \mathrm{e}^{\rho' r_0}(\kappa\lambda_1' + \lambda_2')$. By Gronwall's inequality and (2.3), one has

$$\|X_t\|_\infty^2 \le c_3(1+\|\xi\|_\infty^2) + c_3\mathrm{e}^{-\rho' t}N(t) + c_3\int_0^t \mathrm{e}^{-\rho' s-\lambda'(t-s)}N(s)\mathrm{d}s$$

for some constant $c_3 > 0$, where $\lambda' > 0$ is defined in (2.3). Hence, for any $\varepsilon > 0$, we have

$$\mathbb{E}\mathrm{e}^{\varepsilon\|X_t\|_\infty^2} \le \mathrm{e}^{c_3(1+\|\xi\|_\infty^2)}\sqrt{I_1 \times I_2},$$

in which

$$I_1 := \mathbb{E}\exp\left[2c_3\varepsilon\int_0^t \mathrm{e}^{-\rho' s-\lambda'(t-s)}N(s)\mathrm{d}s\right] \quad \text{and} \quad I_2 := \mathbb{E}\exp\left[2c_3\varepsilon\mathrm{e}^{-\rho' t}N(t)\right].$$

Next, on following arguments of [2, (2.4) and (2.5)], there exists $\varepsilon > 0$ and $c_4 > 0$ such that

$$\mathrm{e}^{\lambda' t}\mathbb{E}\mathrm{e}^{\varepsilon\|X_t\|_\infty^2} \le \mathrm{e}^{c_4(1+\|\xi\|_\infty^2)+\lambda' t} + \frac{\lambda'}{2}\int_{-r_0}^t \mathrm{e}^{\lambda' s}\mathbb{E}\mathrm{e}^{\varepsilon\|X_s\|_\infty^2}\mathrm{d}s.$$

Consequently, an application of the Gronwall inequality gives that

$$\mathbb{E}\mathrm{e}^{\varepsilon\|X_t\|_\infty^2} \le \mathrm{e}^{c_4(1+\|\xi\|_\infty^2)} + \frac{\lambda'}{2}\int_{-r_0}^t \mathrm{e}^{c_4(1+\|\xi\|_\infty^2)}\mathrm{e}^{-\frac{\lambda'(t-s)}{2}}\mathrm{d}s,$$

and the desired assertion (2.1) follows immediately due to $\lambda' > 0$.

□

Remark 2.1. In Lemma 2.2, if $\kappa = 0$, then the first condition in (1.4) is definitely true, and the second one reduces to $\lambda_1 > \lambda_2\mathrm{e}^{\lambda_2 r_0}$, which is imposed in [2, Lemma 2.1] to show the Gauss-type concentration property of the unique invariant probability measure for a range of non-degenerate functional SDEs.

Remark 2.2. By an inspection of argument of Lemma 2.2, we observe that Lemma 2.2 still holds for a general neutral term $G : \mathscr{C} \mapsto \mathbb{R}^n$ provided that $|G(\xi)| \le \kappa\|\xi\|_\infty$ for $\xi \in \mathscr{C}$ and $\kappa \in (0,1)$. However, in Lemma 2.2, we only consider the linear case $G(\xi) = L\xi, \xi \in \mathscr{C}$, this is for the consistency with the Harnack inequality to be established in Lemma 2.5 below.

The lemma below states that segment processes with different initial datum close to each other when the time tends to infinity.

Lemma 2.3. Under the assumptions of Theorem 1.1, there exists a constant $c > 0$ such that

$$\|X_t(\xi) - X_t(\eta)\|_\infty^2 \le c\mathrm{e}^{-\lambda t}\|\xi-\eta\|_\infty^2, \tag{2.11}$$

where $\lambda > 0$ is given in (1.4).

Proof. For simplicity, set

$$\Phi(t) := X(t;\xi) - X(t;\eta) + L(X_t(\xi) - X_t(\eta)).$$

By the chain rule, for $\rho = \lambda_1/(1+\kappa)$, it follows that

$$\mathrm{d}(\mathrm{e}^{\rho t}|\Phi(t)|^2) \le \mathrm{e}^{\rho t}\{\rho|\Phi(t)|^2 - \lambda_1|X(t;\xi) - X(t;\eta)|^2 + \lambda_2\|X_t(\xi) - X_t(\eta)\|_\infty^2\}\mathrm{d}t.$$

Thus, carrying out a similar argument to derive (2.10), we can deduce that

$$\mathrm{e}^{\rho t}\|X_t(\xi) - X_t(\eta)\|_\infty^2 \le C_1\|\xi - \eta\|_\infty^2 + (\rho - \lambda)\int_0^t \mathrm{e}^{\rho s}\|X_s(\xi) - X_s(\eta)\|_\infty^2 \mathrm{d}s$$

for some constant $C_1 > 0$, where $\lambda > 0$ is defined as in (1.4). Next, by virtue of Gronwall's inequality, one finds

$$\|X_t(\xi) - X_t(\eta)\|_\infty^2 \le C_1\mathrm{e}^{-\lambda t}\|\xi - \eta\|_\infty^2.$$

The proof is, therefore, complete. □

Lemma 2.4. Let the conditions of Theorem 1.1 hold. Then, the Markov semigroup P_t, defined in (1.3), admits a unique invariant probability measure $\mu \in \mathscr{P}(\mathscr{C})$.

Proof. The method of the proof is similar to that of Lemma 2.4 in [2], for the convenience of the reader, we give a sketch of the proof. Let W be the L^1-Wasserstein distance induced by the distance $\rho(\xi,\eta) := 1 \wedge \|\xi - \eta\|_\infty$; that is

$$W(\mu_1,\mu_2) := \inf_{\pi \in \mathscr{C}(\mu_1,\mu_2)} \pi(\rho), \quad \mu_1,\mu_2 \in \mathscr{P}(\mathscr{C}),$$

where $\mathscr{C}(\mu_1,\mu_2)$ is the set of all couplings of μ_1 and μ_2. It is well known that $\mathscr{P}(\mathscr{C})$ is a complete metric space with respect to the distance W (see, e.g., [6, Lemma 5.3 and Lemma 5.4]), and the convergence in W is equivalent to the weak convergence whenever ρ is bounded (see, e.g., [6, Theorem 5.6]). Let $\mathbb{P}_t^\xi$ be the law of $X_t(\xi)$. So, to show existence of an invariant measure, it is sufficient to claim that $\mathbb{P}_t^\xi$ is a W-Cauchy sequence, i.e.,

$$\lim_{t_1,t_2\to\infty} W(\mathbb{P}_{t_1}^\xi, \mathbb{P}_{t_2}^\xi) = 0. \tag{2.12}$$

For any $t_2 > t_1 > 0$, consider the following neutral SDEs of neutral type

$$\mathrm{d}\{X(t) + LX_t\} = \{Z(X(t)) + b(X_t)\}\mathrm{d}t + \sigma\mathrm{d}W(t),\ t \in [0,t_2], \quad X_0 = \xi,$$

and

$$\mathrm{d}\{\overline{X}(t) + L\overline{X}_t\} = \{Z(\overline{X}(t)) + b(\overline{X}_t)\}\mathrm{d}t + \sigma\mathrm{d}W(t),\ t \in [t_2 - t_1, t_2], \quad \overline{X}_{t_2-t_1} = \xi.$$

Then the laws of $X_{t_2}(\xi)$ and $\overline{X}_{t_2}(\xi)$ are $\mathbb{P}_{t_2}^\xi$ and $\mathbb{P}_{t_1}^\xi$, respectively. Also, following an argument of (2.10), we can obtain that

$$\mathrm{e}^{\rho t}\mathbb{E}\|X_t - \overline{X}_t\|_\infty^2 \le C_2\mathbb{E}\|X_{t_2-t_1} - \xi\|_\infty^2 + (\rho - \lambda)\int_{t_2-t_1}^t \mathrm{e}^{\rho s}\mathbb{E}\|X_s - \overline{X}_s\|_\infty^2 \mathrm{d}s$$

for some constant $C_2>0$, in which $\lambda>0$ is defined as in (1.4). According to the Gronwall inequality, one finds

$$\mathbb{E}\|X_t-\overline{X}_t\|_\infty^2\le C_2\mathrm{e}^{-\lambda(t-t_2+t_1)}\mathbb{E}\|X_{t_2-t_1}-\xi\|_\infty^2.$$

This, in addition to (2.1), gives that

$$\mathbb{E}\|X_{t_2}-\overline{X}_{t_1}\|_\infty^2\le C_2\mathrm{e}^{-\lambda t_1},$$

which further implies

$$W(\mathbb{P}_{t_1}^\xi,\mathbb{P}_{t_2}^\xi)\le\mathbb{E}\|X_{t_2}-\overline{X}_{t_1}\|_\infty\le\sqrt{C_2}\mathrm{e}^{-\frac{\lambda t_1}{2}}.$$

Then, (2.12) holds by taking $t_1\to\infty$. So, there exists $\mu^\xi\in\mathscr{P}(\mathscr{C})$ such that

$$\lim_{t\to\infty}W(\mathbb{P}_t^\xi,\mu^\xi)=0. \tag{2.13}$$

Thus, the desired assertion follows provided that we can show that μ^ξ is independent of $\xi\in\mathscr{C}$. To this end, note that

$$W(\mu^\xi,\mu^\eta)\le W(\mathbb{P}_t^\xi,\mu^\xi)+W(\mathbb{P}_t^\eta,\mu^\eta)+W(\mathbb{P}_t^\xi,\mathbb{P}_t^\eta).$$

Taking $t\to\infty$ and using (2.11) and (2.13), we conclude that $\mu^\xi\equiv\mu^\eta$ for any $\xi,\eta\in\mathscr{C}$. Hence, we conclude that μ^ξ is independent of $\xi\in\mathscr{C}$. □

The condition (i) in Lemma 2.1 is concerned with the dimension-free Harnack inequality which is initiated in [15]. To the best of our knowledge, coupling by change of measure (see, e.g., the monograph [19]) and Malliavin calculus (see, e.g., [4]) are two popular approaches to establish the Harnack inequality, which has considerable applications in contractivity properties, functional inequalities, short-time behaviors of infinite-dimensional diffusions, as well as heat kernel estimates (see, e.g., Wang [19, 17]). For the Harnack inequality of stochastic partial differential equations (SPDEs), we refer to the monograph [19]. For the Harnack inequality of SDEs with memory, we would like to refer to [8] for functional SDEs with additive noises, [14] for functional SDEs with multiplicative noises, [3] for stochastic functional Hamiltonian systems with degenerate noises, and [4] for functional SPDEs with additive noises.

In this paper, we also adopt the coupling by change of measure (see, e.g., the monograph [19]) to establish the Harnack inequality for (1.1). However, due to the appearance of neutral term, as we observe below, it becomes more tricky to investigate the Harnack inequality for the functional SDE of neutral type (1.1).

For $\xi_x(\theta)\equiv x$ and $\eta_y(\theta)\equiv y$ with $\theta\in[-r_0,0]$, from (H1) and (H2), there exists $\kappa_1\in\mathbb{R}$ such that

$$\langle Z(x)-Z(y),x-y\rangle\le-\kappa_1|x-y|^2,\quad x,y\in\mathbb{R}^n. \tag{2.14}$$

Lemma 2.5. Let the assumptions of Theorem 1.1 hold. Then, there exists a constant $c > 0$ such that

$$(P_t f(\xi))^2 \le (P_t f^2(\eta))\mathrm{e}^{c\|\xi-\eta\|_\infty^2}, \quad f \in \mathscr{B}_b(\mathscr{C}),\ \xi,\ \eta \in \mathscr{C},\ t > r_0. \tag{2.15}$$

Proof. We adopt the coupling by change of measures (see, e.g., the monograph [19]) to establish the Harnack inequality (2.15). Let $\{Y(s)\}_{s\ge 0}$ solve an SDE without memory

$$\begin{aligned} &\mathrm{d}\{Y(s) + LX_s\} \\ &= \Big\{Z(Y(s)) + b(X_s) + g(s)1_{[0,\tau)}(s) \cdot \frac{X(s) - Y(s)}{|X(s) - Y(s)|}\Big\}\mathrm{d}s + \sigma \mathrm{d}W(s), \quad s > 0, \end{aligned} \tag{2.16}$$

with $Y_0 = \eta \in \mathscr{C}$, where

$$\tau := \inf\{s \ge 0 : X(s) = Y(s)\}$$

is the coupling time and $g : [0,\infty) \mapsto \mathbb{R}_+$ is a continuous mapping to be determined. By (2.14) and the chain rule,

$$\begin{aligned} \mathrm{d}(\mathrm{e}^{\kappa_1 s}|X(s) - Y(s)|) &= \kappa_1 \mathrm{e}^{\kappa_1 s}|X(s) - Y(s)|\mathrm{d}s + \mathrm{e}^{\kappa_1 s}\{-g(s)1_{[0,\tau)}(s) \\ &\quad + |X(s) - Y(s)|^{-1}\langle X(s) - Y(s), Z(X(s)) - Z(Y(s))\rangle\}\mathrm{d}s \\ &\le -\mathrm{e}^{\kappa_1 s}g(s)\mathrm{d}s, \qquad s < \tau. \end{aligned}$$

Thus, one has

$$|X(s) - Y(s)| \le \mathrm{e}^{-\kappa_1 s}|\xi(0) - \eta(0)| - \mathrm{e}^{-\kappa_1 s}\int_0^s \mathrm{e}^{\kappa_1 r}g(r)\mathrm{d}r, \quad s \le \tau. \tag{2.17}$$

In (2.17), in particular, choosing

$$g(r) = \frac{|\xi(0) - \eta(0)|\mathrm{e}^{\kappa_1 r}}{\int_0^t \mathrm{e}^{2\kappa_1 r}\mathrm{d}r}, \quad r \in [0,t] \tag{2.18}$$

leads to

$$|X(s) - Y(s)| \le \frac{|\xi(0) - \eta(0)|(\mathrm{e}^{2\kappa_1 t - \kappa_1 s} - \mathrm{e}^{\kappa_1 s})}{\mathrm{e}^{2\kappa_1 t} - 1}, \quad s \le \tau. \tag{2.19}$$

If $t < \tau$, (2.19) implies $X(t) = Y(t)$, which contradicts the definition of coupling time τ. Consequently, we have $\tau \le t$. Also, by the chain rule, for any $\varepsilon > 0$, we obtain from (2.14) and $X(\tau) = Y(\tau)$ that

$$\begin{aligned} (\varepsilon + |X(t) - Y(t)|^2)^{1/2} &= (\varepsilon + |X(\tau) - Y(\tau)|^2)^{1/2} \\ &\quad + \int_\tau^t (\varepsilon + |X(s) - Y(s)|^2)^{-1/2}\langle X(s) - Y(s), Z(X(s)) - Z(Y(s))\rangle\mathrm{d}s \\ &\le \sqrt{\varepsilon} - \kappa_1\int_\tau^t (\varepsilon + |X(s) - Y(s)|^2)^{-1/2}(\varepsilon + |X(s) - Y(s)|^2 - \varepsilon)\mathrm{d}s \\ &\le (1 + |\kappa_1|(t-\tau))\sqrt{\varepsilon} + |\kappa_1|\int_\tau^t (\varepsilon + |X(s) - Y(s)|^2)^{1/2}\mathrm{d}s, \quad t > \tau. \end{aligned}$$

Taking $\varepsilon \downarrow 0$ and utilizing Gronwall's inequality, we conclude that $X(t) = Y(t)$ for any $t \geq \tau$. Hence, $X_{t+r_0} = Y_{t+r_0}$, and

$$|X(s) - Y(s)| \leq \frac{|\xi(0) - \eta(0)|(\mathrm{e}^{2\kappa_1 t - \kappa_1 s} - \mathrm{e}^{\kappa_1 s})}{\mathrm{e}^{2\kappa_1 t} - 1} =: G(s), \quad s \leq t. \tag{2.20}$$

Let

$$\widetilde{W}(s) := \int_0^s \sigma^{-1} h(r)\mathrm{d}r + W(s), \quad s \in [0, t + r_0],$$

where

$$\begin{aligned} h(r) : &= \{\xi(r - r_0) - \eta(r - r_0) + Y(r) - X(r)\} 1_{[0,r_0]}(r) + \Big(\int_{r-r_0}^r \Lambda(u)\mathrm{d}u\Big) 1_{(r_0,t+r_0]}(r) \\ &\quad + \Big\{1_{[0,\tau)} g(r) \frac{X(r) - Y(r)}{|X(r) - Y(r)|} + b(X_r) - b(Y_r)\Big\} \\ &=: h_1(r) 1_{[0,r_0]}(r) + h_2(r) 1_{(r_0,t+r_0]}(r) + h_3(r). \end{aligned}$$

Herein $\Lambda(u) := Z(Y(u)) - Z(X(u)) + g(u) 1_{[0,\tau)}(u) \cdot \frac{X(u)-Y(u)}{|X(u)-Y(u)|}$, $u \in [s - r_0, s]$. For any $s \in [0, r_0]$, note that

$$\begin{aligned} &\int_0^s h_1(r)\mathrm{d}r + \int_0^{r_0} \{\eta(u - r_0) - \xi(u - r_0)\}\mathrm{d}u \\ &= \int_{-r_0}^{-s} \{\eta(s + \theta) - \xi(s + \theta)\}\mathrm{d}\theta + \int_{-s}^0 \{Y(s + \theta) - X(s + \theta)\}\mathrm{d}\theta \\ &= \int_{-r_0}^0 \{\eta(s + \theta) - \xi(s + \theta)\} 1_{\{s+\theta \leq 0\}}\mathrm{d}\theta + \int_{-r_0}^0 \{Y(s + \theta) - X(s + \theta)\} 1_{\{s+\theta > 0\}}\mathrm{d}\theta \\ &= L(Y_s - X_s). \end{aligned} \tag{2.21}$$

From (1.1) and (2.16), it is trivial to see that

$$Y(r) - X(r) = \eta(0) - \xi(0) + \int_0^r \Lambda(u)\mathrm{d}u, \quad r \geq 0. \tag{2.22}$$

For arbitrary $s \in (r_0, t + r_0]$, according to (2.22), it follows that

$$\begin{aligned} &(\eta(0) - \xi(0))r_0 + \int_0^{r_0} (r_0 - u)\Lambda(u)\mathrm{d}u + \int_{r_0}^s h_2(r)\mathrm{d}r \\ &= (\eta(0) - \xi(0))r_0 + \int_0^{r_0} (r_0 - u)\Lambda(u)\mathrm{d}u \\ &\quad + \int_{r_0}^s \Big(r_0 \int_0^{r-r_0} \Lambda(u)\mathrm{d}u + \int_{r-r_0}^r (r - u)\Lambda(u)\mathrm{d}u\Big)' \mathrm{d}r \\ &= (\eta(0) - \xi(0))r_0 + r_0 \int_0^{s-r_0} \Lambda(u)\mathrm{d}u + \int_{s-r_0}^s (s - u)\Lambda(u)\mathrm{d}u \\ &= \int_{-r_0}^0 \Big\{\eta(0) - \xi(0) + \int_0^{s+\theta} \Lambda(u)\mathrm{d}u\Big\}\mathrm{d}\theta \\ &= L(Y_s - X_s). \end{aligned} \tag{2.23}$$

Hence, from (2.21) and (2.23), we arrive at

$$\mathrm{d}L(Y_s - X_s) = \{h_1(s)1_{[0,r_0]}(s) + h_2(s)1_{(r_0,t+r_0]}(s)\}\mathrm{d}s, \qquad s \in [0, r_0 + t]. \tag{2.24}$$

Observing that $G(s)$, defined in (2.20), is decreasing for $s \in [0,t]$ and taking (2.20) into account gives that

$$\|X_s - Y_s\|_\infty^2 \le 1_{[0,r_0]}(s)\|\xi - \eta\|_\infty^2 + 1_{(r_0,t+r_0]}(s)\frac{|\xi(0)-\eta(0)|^2(\mathrm{e}^{2\kappa_1 t - \kappa_1(s-r_0)} - \mathrm{e}^{\kappa_1(s-r_0)})^2}{(\mathrm{e}^{2\kappa_1 t}-1)^2}. \tag{2.25}$$

By (H1), (2.20) and (2.25), for any $\alpha, \beta, \delta > 0$, we derive from Hölder's inequality that

$$\begin{aligned}
&|h(s)|^2 \\
&\le (1+\alpha^{-1})\Big\{4\|\xi-\eta\|_\infty^2 1_{[0,r_0]}(s) + r_0\int_{s-r_0}^t \Big((1+\delta)1_{[0,t]}(u)g^2(u) \\
&\quad + (1+\delta^{-1})L_1^2|X(u)-Y(u)|^2\Big)\mathrm{d}u 1_{(r_0,t+r_0]}(s) \\
&\quad + r_0\int_t^s \Big((1+\delta)1_{[0,t]}(u)g^2(u) \\
&\quad + (1+\delta^{-1})L_1^2|X(u)-Y(u)|^2\Big)\mathrm{d}u 1_{(r_0,t]}(s)\Big\} \\
&\quad + (1+\alpha)\{(1+\beta)g^2(s)1_{[0,t]}(s) + (1+\beta^{-1})L_2^2\|X_s-Y_s\|_\infty^2\} \\
&\le (1+\alpha^{-1})\Big\{4\|\xi-\eta\|_\infty^2 1_{[0,r_0]}(s) + r_0\int_{s-r_0}^t \Big((1+\delta)1_{[0,t]}(u)g^2(u) \\
&\quad + (1+\delta^{-1})L_1^2|X(u)-Y(u)|^2\Big)\mathrm{d}u 1_{(r_0,t+r_0]}(s)\Big\} \\
&\quad + (1+\alpha)\{(1+\beta)g^2(s)1_{[0,t]}(s) + (1+\beta^{-1})L_2^2\|X_s-Y_s\|_\infty^2\} \\
&\le (1+\alpha^{-1})\Big\{4\|\xi-\eta\|_\infty^2 1_{[0,r_0]}(s) \\
&\quad + \frac{2r_0\kappa_1(1+\delta)|\xi(0)-\eta(0)|^2}{(\mathrm{e}^{2\kappa_1 t}-1)^2}\Big(\mathrm{e}^{2\kappa_1 t} - \mathrm{e}^{2\kappa_1(s-r_0)}\Big)1_{(r_0,t+r_0]}(s) \\
&\quad + r_0 L_1^2(1+\delta^{-1})(t-s+r_0)G^2(s-r_0)1_{(r_0,t+r_0]}(s)\Big\} \\
&\quad + (1+\alpha)\Big\{\frac{4(1+\beta)\kappa_1^2|\xi(0)-\eta(0)|^2\mathrm{e}^{2\kappa_1 s}}{(\mathrm{e}^{2\kappa_1 t}-1)^2}1_{[0,t]}(s) \\
&\quad + (1+\beta^{-1})L_2^2\|\xi-\eta\|_\infty^2 1_{[0,r_0]}(s) + (1+\beta^{-1})L_2^2 G^2(s-r_0)1_{(r_0,t+r_0]}(s)\Big\}.
\end{aligned} \tag{2.26}$$

This, together with a straightforward calculation, yields

$$\mathbb{E}\exp\Big(\frac{1}{2}\int_0^{t+r_0}|\sigma^{-1}h(r)|^2\mathrm{d}r\Big) < \infty.$$

As a result, Novikov's condition holds so that, by the Girsanov theorem, $\{\widetilde{W}(s)\}_{t\in[0,t+r_0]}$ is a Brownian motion under the weighted probability measure $\mathrm{d}\mathbb{Q} := R\mathrm{d}\mathbb{P}$ with

$$R := \exp\Big[-\int_0^{t+r_0}\langle\sigma^{-1}h(s), \mathrm{d}W(s)\rangle - \frac{1}{2}\int_0^{t+r_0}|\sigma^{-1}h(s)|^2\mathrm{d}s\Big].$$

Due to (2.24), equation (2.16) can be reformulated as

$$\mathrm{d}\Big\{Y(s)+\int_{-\tau}^{0}Y(s+\theta)\nu(\mathrm{d}\theta)\Big\}=\{Z(Y(s))+b(Y_s)\}\mathrm{d}s+\sigma\mathrm{d}\widetilde{W}(s)$$

with the initial data $Y_0=\eta\in\mathscr{C}$. By virtue of weak uniqueness of solutions and $X_{t+r_0}(\xi)=Y_{t+r_0}(\eta)$, in addition to the Hölder inequality, one finds,

$$\begin{aligned}(P_{t+r_0}f(\eta))^2&=(\mathbb{E}f(X_{t+r_0}(\eta)))^2=(\mathbb{E}_{\mathbb{Q}}f(Y_{t+r_0}(\eta)))^2=(\mathbb{E}[Rf(Y_{t+r_0}(\eta))])^2\\&=(\mathbb{E}[Rf(X_{t+r_0}(\xi))])^2\le\mathbb{E}R^2P_{t+r_0}f^2(\xi),\end{aligned}\tag{2.27}$$

and that

$$\begin{aligned}&\mathbb{E}R^2\\&\le\Big(\mathbb{E}\mathrm{e}^{6\int_0^{t+r_0}|\sigma^{-1}h(s)|^2\mathrm{d}s}\Big)^{1/2}\Big(\mathbb{E}\mathrm{e}^{-4\int_0^{t+r_0}\langle\sigma^{-1}h(s),\mathrm{d}B(s)\rangle-8\int_0^{t+r_0}|\sigma^{-1}h(s)|^2\mathrm{d}s}\Big)^{1/2}\\&=\Big(\mathbb{E}\mathrm{e}^{6\int_0^{t+r_0}|h(s)|^2\mathrm{d}s}\Big)^{1/2}.\end{aligned}\tag{2.28}$$

Hence, from (2.26)-(2.28), for any $t_0>r_0,p>1,\alpha,\beta,\delta>0$, positive $f\in\mathscr{B}_b(\mathscr{C})$, and $\xi,\eta\in\mathscr{C}$, we deduce that

$$\begin{aligned}&(P_{t_0}f(\eta))^2\\&\le P_{t_0}f^2(\xi)\exp\bigg[3\|\sigma^{-1}\|^2\Big\{(1+\alpha^{-1})\Big(4\|\xi-\eta\|_\infty^2r_0\\&\quad+\frac{r_0(1+\delta)|\xi(0)-\eta(0)|^2}{(\mathrm{e}^{2\kappa_1t}-1)^2}\Big(2\kappa_1t\mathrm{e}^{2\kappa_1t}+1-\mathrm{e}^{2\kappa_1t}\Big)\\&\quad+\frac{r_0L_1^2(1+\delta^{-1})|\xi(0)-\eta(0)|^2}{4\kappa_1^2(\mathrm{e}^{2\kappa_1t}-1)^2}\times\Big((2\kappa_1t-1)(\mathrm{e}^{4\kappa_1t}-4\kappa_1t\mathrm{e}^{2\kappa_1t}-1)\\&\quad+2(\mathrm{e}^{2\kappa_1t}-1)+4\kappa_1t\mathrm{e}^{2\kappa_1t}(2\kappa_1-1)\Big)\Big)\\&\quad+(1+\alpha)\Big(\frac{4(1+\beta)\kappa_1|\xi(0)-\eta(0)|^2}{2(\mathrm{e}^{2\kappa_1t}-1)}\\&\quad+(1+\beta^{-1})L_2^2r_0\|\xi-\eta\|_\infty^2+\frac{(1+\beta^{-1})L_2^2|\xi(0)-\eta(0)|^2(\mathrm{e}^{4\kappa_1t}-4\kappa_1t\mathrm{e}^{2\kappa_1t}-1)}{2\kappa_1(\mathrm{e}^{2\kappa_1t}-1)^2}\Big)\Big\}.\end{aligned}\tag{2.29}$$

By the Markov property and Schwartz's inequality, for any $t>0$, we deduce from (2.29) that there exists $c_0>0$ such that

$$\begin{aligned}|P_{t+t_0}f(\xi)|^2=|\mathbb{E}(P_{t_0}f)(X_t(\xi))|^2&\le\Big(\mathbb{E}\sqrt{(P_{t_0}f^2(X_t(\eta)))\exp[c_0\|X_t(\xi)-X_t(\eta)\|_\infty^2]}\Big)^2\\&\le(\mathbb{E}(P_{t_0}f^2(X_t(\eta)))\mathbb{E}\mathrm{e}^{c_0\|X_t(\xi)-X_t(\eta)\|_\infty^2}\\&=(P_{t+t_0}f^2(\eta))\mathbb{E}\mathrm{e}^{c_0\|X_t(\xi)-X_t(\eta)\|_\infty^2}.\end{aligned}$$

At last, the desired assertion follows from (2.11) immediately.

□

Remark 2.3. In Lemma 2.5, we only investigate the Harnack inequality for a special class of functional SDEs of neutral type. It is still open on how to establish the Harnack inequality for functional SDEs of neutral type with general neutral terms.

Proof of Theorem 1.1 By virtue of Lemma 2.1, Theorem 1.1 (1)-(3) follows from Lemma 2.2, Lemma 2.4, and Lemma 2.5. According to [20, Proposition 2.3], Theorem 1.1 (1) implies the desired exponential convergence of P_t in entropy and in $L^2(\mu)$ in Theorem 1.1 (3) and (4), respectively. Moreover, Theorem 1.1 (5) follows by carrying out a similar argument to that of [2, Theorem 1.1 (4)].

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